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The Dyadic KMS Spectrum of Arithmetic Diamond Matrices and the $3 \bmod 4$ Prime-Power Phase Transition in Shifted Smith Determinants

Abstract. In 1875, H. J. S. Smith established that the determinant of the $n \times n$ greatest common divisor matrix $G_n = [\gcd(i,j)]_{1\le i,j\le n}$ is given by $\det(G_n) = \prod_{k=1}^{n} \phi(k)$. Smith's proof relies fundamentally on the fact that the index set $\{1, 2, \ldots, n\}$ is factor-closed. In this monograph, we investigate two fundamental deformations of Smith's matrix that break factor-closure by coupling multiplicative divisibility with additive operations:

1. **The Arithmetic Diamond Matrix** $A_n = \big[\gcd(i+j,\, |i-j|)\big]_{1\le i,j\le n}$ and its general $(s,\lambda;p)$-deformation $A_n^{(s,\lambda;p)} = \big[\gcd(i,j)^s(1+(\lambda-1)\delta_{v_p(i),v_p(j)})\big]_{1\le i,j\le n}$. We prove that A_n admits an exact block-diagonal congruence over the odd integers into dyadic Kac–Murdock–Szegő (KMS) Toeplitz operators at the irrational coupling $\rho = 1/\sqrt{2}$. Consequently, while the step-by-step determinant ratios $r_n = \det(A_n)/\det(A_{n-1})$ exhibit a cascade of "ghost primes" $(7, 41, 239, 1393, 8119, \ldots)$ that appear at scale $n = 2^v m$ and annihilate at scale $n = 2^{v+1}m'$, the cumulative determinant $\det(A_n)$ is governed in exact closed form by the **Silver Ratio** $\sigma = 1+\sqrt{2}$, the **Pell numbers** P_k, and the **Companion Pell numbers** Q_k.
2. **The Additively Shifted GCD Matrix** $G_n^{(k)} = \big[\gcd(k+i,\, k+j)\big]_{1\le i,j\le n}$ over all integer shifts $k \in \mathbb{Z}_{\ge 0}$. Computational exploration for $2 \le n \le 13$ suggests the universal law $\gcd_{k\ge 0} \det G_n^{(k)} = 2^{n-2}$, with $\min_{k\ge 0} v_2(\det G_n^{(k)}) = n-2$. We prove that $2^{n-2} \mid \det G_n^{(k)}$ holds universally for all $n \ge 2$ and all $k \in \mathbb{Z}$, **but that the equality $\min_{k\ge 0} v_2(\det G_n^{(k)}) = n-2$ fails for the first time at the critical dimension $n = 14$**. By constructing a two-stage 2-adic filtration over $\mathbb{F}_2$, we prove that the excess 2-adic valuation is governed by a bottleneck between the dimension of the mod-4 even difference space $\lfloor n/2 \rfloor - 2$ and the counting function $\pi^*_{3,4}(n-1)$ of odd prime powers $p^r \le n-1$ with $p \equiv 3 \pmod 4$:

$$\min_{k\ge 0} v_2\big(\det G_n^{(k)}\big) \ge (n-2) + \max\Big(0,\, \lfloor n/2 \rfloor - 2 - \pi^*_{3,4}(n-1)\Big).$$

Part I · The Arithmetic Diamond Matrix and the Silver-Ratio KMS Law

1.1 The Ghost-Prime Phenomenon

Define the $n \times n$ **Arithmetic Diamond Matrix** by coupling the additive sum $i+j$ and difference $|i-j|$ inside the greatest common divisor:

$$A_n := \Big[\gcd(i+j,\, |i-j|)\Big]_{1\le i,j\le n}.$$

When we compute $\det(A_n)$ and the consecutive determinant ratios $r_n := \frac{\det(A_n)}{\det(A_{n-1})}$ (with $\det(A_0) := 1$), a startling arithmetic phenomenon emerges:

n	$n = 2^v \cdot m$	$\det(A_n)$	**Prime Factorization of** $\det(A_n)$	**Step Ratio** $r_n = \frac{\det(A_n)}{\det(A_{n-1})}$	**Normalized Ratio** $\frac{r_n}{\phi(m)}$
1	$2^0 \cdot 1$	2	2	2	2
2	$2^1 \cdot 1$	7	**7**	7/2	7/2
3	$2^0 \cdot 3$	28	$2^2 \cdot$ **7**	4	2
4	$2^2 \cdot 1$	192	$2^6 \cdot 3$ *(the* 7 *vanishes!)*	48/7	48/7
5	$2^0 \cdot 5$	1536	$2^9 \cdot 3$	8	2
6	$2^1 \cdot 3$	10752	$2^9 \cdot 3 \cdot 7$	7	7/2
7	$2^0 \cdot 7$	129024	$2^{11} \cdot 3^2 \cdot 7$	12	2
8	$2^3 \cdot 1$	1763328	$2^{11} \cdot 3 \cdot 7 \cdot$ **41**	41/3	41/3

n	$n = 2^v \cdot m$	$\det(A_n)$	**Prime Factorization of** $\det(A_n)$	**Step Ratio** $r_n = \frac{\det(A_n)}{\det(A_{n-1})}$	**Normalized Ratio** $\frac{r_n}{\phi(m)}$
9	$2^0 \cdot 9$	21159936	$2^{13} \cdot 3^2 \cdot 7 \cdot \mathbf{41}$	12	2
10	$2^1 \cdot 5$	296239104	$2^{14} \cdot 3^2 \cdot 7^2 \cdot \mathbf{41}$	14	7/2
11	$2^0 \cdot 11$	5924782080	$2^{16} \cdot 3^2 \cdot 5 \cdot 7^2 \cdot \mathbf{41}$	20	2
12	$2^2 \cdot 3$	81254154240	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot \mathbf{41}$	96/7	48/7
16	$2^4 \cdot 1$	17899061457715200	$2^{33} \cdot 3^5 \cdot 5^2 \cdot 7^3$ *(the* 41 *vanishes!)*	1120/41	1120/41
32	$2^5 \cdot 1$	—	$\cdots \times \mathbf{239}$	1912/35	1912/35
64	$2^6 \cdot 1$	—	*(the* 239 *vanishes!)*	26112/239	26112/239

> [!IMPORTANT] **Two Observations from the Table:**
>
> 1. Writing $n = 2^v m$ with m odd and $v = v_2(n)$, the normalized step ratio $c_v := \frac{r_{2^v m}}{\phi(m)}$ is **completely independent of the odd part** m:
>
> $$c_0 = 2, \quad c_1 = \frac{7}{2}, \quad c_2 = \frac{48}{7}, \quad c_3 = \frac{41}{3}, \quad c_4 = \frac{1120}{41}, \quad c_5 = \frac{1912}{35}, \quad c_6 = \frac{26112}{239}, \quad c_7 = \frac{11144}{51}.$$
>
> 2. At every odd power of two ($n = 2^1, 2^3, 2^5, 2^7, \ldots$), a prime divisor ($7, 41, 239, 1393, 8119, \ldots$) is injected into $\det(A_n)$, only to be **exactingly cancelled** in the denominator of c_{v+1} at the next power of two ($n = 2^2, 2^4, 2^6, 2^8, \ldots$). These are the **Companion Pell numbers** Q_{2k+1}, the numerators of the continued fraction convergents to $\sqrt{2}$.

1.2 Exact 2-Adic Separation of the Diamond Kernel

The bridge between $\gcd(i+j, |i-j|)$ and the 2-adic valuation v_2 is given by the following exact arithmetic identity.

Lemma 1.1 (2-Adic Resonance Identity)

For all positive integers $i, j \in \mathbb{Z}_{\geq 1}$,

$$\gcd(i+j,\, |i-j|) = \gcd(i,j) \cdot \big(1 + \delta_{v_2(i), v_2(j)}\big),$$

where $\delta_{a,b} = 1$ if $a = b$ and 0 otherwise.

Proof. Let $d = \gcd(i,j)$, and write $i = dx$, $j = dy$ with $\gcd(x,y) = 1$. By homogeneity of the greatest common divisor,

$$\gcd(i+j,\, |i-j|) = d \cdot \gcd(x+y,\, |x-y|).$$

Since $(x+y) + (x-y) = 2x$ and $(x+y) - (x-y) = 2y$, any common divisor of $x+y$ and $x-y$ divides $\gcd(2x, 2y) = 2\gcd(x,y) = 2$. Conversely, $\gcd(x+y, |x-y|) \equiv x + y \pmod 2$. Since $\gcd(x,y) = 1$, x and y cannot both be even:

1. If $v_2(i) \neq v_2(j)$, then one of x, y is even and the other is odd, so $x+y$ is odd and $\gcd(x+y, |x-y|) = 1$.
2. If $v_2(i) = v_2(j)$, then both x and y are odd, so $x+y$ and $x-y$ are both even, and $\gcd(x+y, |x-y|) = 2$. ■

1.3 Odd-Divisor Congruence and the General $(s, \lambda; p)$-Diamond Operator

Motivated by Lemma 1.1, let p be any prime, $s \in \mathbb{C}$, and $\lambda \in \mathbb{C}$, and define the **Generalized $(s, \lambda; p)$-Arithmetic Diamond Matrix** of order n:

$$A_n^{(s,\lambda;p)}(i,j) := \gcd(i,j)^s \Big(1 + (\lambda - 1)\delta_{v_p(i), v_p(j)}\Big) \qquad (1 \leq i, j \leq n).$$

When $p = 2, s = 1, \lambda = 2$, we recover $A_n^{(1,2;2)} = \big[\gcd(i+j, |i-j|)\big]_{1 \leq i,j \leq n}$. When $p = 2, \lambda = 2$ and $s \in \mathbb{N}$, we obtain the s-power diamond matrix $\big[\gcd(i+j, |i-j|)^s\big]_{1 \leq i,j \leq n}$.

Theorem 1.2 (Odd/Coprime Block-Diagonal Decomposition)

Let p be a prime and let $\mathcal{M}_p(n) := \{1 \le m \le n : p \nmid m\}$ be the set of positive integers up to n coprime to p. For each $m \in \mathcal{M}_p(n)$, let

$$V_p(n,m) := \lfloor \log_p(n/m) \rfloor$$

so that every $i \in \{1,\dots,n\}$ is uniquely written as $i = p^a u$ with $u \in \mathcal{M}_p(n)$ and $0 \le a \le V_p(n,u)$. Then there exists a unit lower-triangular integer matrix L_n ($\det L_n = 1$) such that

$$A_n^{(s,\lambda;p)} = L_n \left(\bigoplus_{m \in \mathcal{M}_p(n)} J_s(m) \cdot D^{(s,\lambda;p)}_{V_p(n,m)} \right) L_n^T,$$

where $J_s(m) = \sum_{d \mid m} \mu(m/d) d^s = m^s \prod_{q \mid m} (1 - q^{-s})$ is Jordan's totient function (with $J_1(m) = \phi(m)$), and $D_V^{(s,\lambda;p)}$ is the $(V+1) \times (V+1)$ p-adic core matrix with entries

$$D_V^{(s,\lambda;p)}(a,b) := p^{s \min(a,b)} \big(1 + (\lambda - 1)\delta_{a,b}\big) \qquad (0 \le a,b \le V).$$

Proof. Write $i = p^a u$ and $j = p^b w$ with $u, w \in \mathcal{M}_p(n)$. Since $\gcd(p^a u, p^b w) = p^{\min(a,b)} \gcd(u,w)$ and $v_p(i) = a, v_p(j) = b$, the matrix entries separate multiplicatively:

$$A_n^{(s,\lambda;p)}(p^a u, p^b w) = D^{(s,\lambda;p)}(a,b) \cdot \gcd(u,w)^s.$$

By Möbius inversion, $\gcd(u,w)^s = \sum_{m \mid \gcd(u,w)} J_s(m)$. Since $p \nmid u$ and $p \nmid w$, every common divisor $m \mid \gcd(u,w)$ lies in $\mathcal{M}_p(n)$. Furthermore, if $m \mid u$ and $p^a u \le n$, then $p^a m \le p^a u \le n$, so the index set is factor-closed with respect to divisors $m \in \mathcal{M}_p(n)$ at each fixed p-adic valuation a. Defining the square matrix $L_n(p^a u, p^b m) := \delta_{a,b} \mathbf{1}_{m \mid u}$, ordering indices lexicographically by (u,a) makes L_n unit lower-triangular ($\det L_n = 1$), and direct matrix multiplication yields

$$\left(L_n \operatorname{diag}\left(J_s(m) D^{(s,\lambda;p)}_{V_p(n,m)}\right) L_n^T\right)_{p^a u,\, p^b w} = \sum_{m \in \mathcal{M}_p(n)} \mathbf{1}_{m \mid u} \mathbf{1}_{m \mid w} J_s(m) D^{(s,\lambda;p)}(a,b) = D^{(s,\lambda;p)}(a,b) \gcd(u,w)^s.$$

Taking determinants immediately yields

$$\det\left(A_n^{(s,\lambda;p)}\right) = \prod_{m \in \mathcal{M}_p(n)} J_s(m)^{V_p(n,m)+1} \det\left(D^{(s,\lambda;p)}_{V_p(n,m)}\right). \quad \blacksquare$$

1.4 Kac–Murdock–Szegő Tridiagonal Reduction of the p-Adic Core

To evaluate $\det\left(D_V^{(s,\lambda;p)}\right)$ in closed form for all $V \ge 0$, we uncover a direct connection to the **Kac–Murdock–Szegő (KMS) Toeplitz matrix** $K_V(\rho) := \left[\rho^{|a-b|}\right]_{0 \le a,b \le V}$.

Theorem 1.3 (The KMS Tridiagonal Reduction & Second-Order Recurrence)

Let $q := p^s$ and define the normalized determinant polynomial

$$\Delta_V(q,\lambda) := \frac{\det\left(D_V^{(s,\lambda;p)}\right)}{q^{V(V-1)/2}} \qquad (V \ge 0).$$

Then:

1. *For all $V \ge 0$, $\Delta_V(q,\lambda) = \frac{1}{q-1} \det \widetilde{W}_V(q,\lambda)$, where $\widetilde{W}_V(q,\lambda)$ is the $(V+1) \times (V+1)$ symmetric tridiagonal matrix*

$$\widetilde{W}_V(q,\lambda) = \begin{pmatrix} \alpha_0 & \beta & 0 & \cdots & 0 \\ \beta & \alpha & \beta & \cdots & 0 \\ 0 & \beta & \alpha & \ddots & \vdots \\ \vdots & \vdots & \ddots & \alpha & \beta \\ 0 & 0 & \cdots & \beta & \alpha_0 \end{pmatrix}_{(V+1)\times(V+1)}$$

with corner diagonal $\alpha_0 = \lambda q - 1$, interior diagonal $\alpha = \lambda q + (\lambda - 2)$, and off-diagonal square $\beta^2 = (\lambda - 1)^2 q$.

2. *The sequence* $\Delta_V = \Delta_V(q,\lambda)$ *satisfies the exact second-order constant-coefficient linear recurrence*

$$\boxed{\Delta_V = \big(\lambda q + \lambda - 2\big)\Delta_{V-1} - (\lambda-1)^2 q\,\Delta_{V-2} \qquad (\forall V \geq 2)}$$

with initial values $\Delta_0 = \lambda$ *and* $\Delta_1 = \lambda^2 q - 1$.

Proof. Let $\rho := q^{-1/2} = p^{-s/2}$ and let $S_V := \operatorname{diag}(1, \rho^{-1}, \rho^{-2}, \dots, \rho^{-V})$. Since $q^{\min(a,b)} = \rho^{-(a+b)}\rho^{|a-b|}$, factoring S_V from both sides of $D_V^{(s,\lambda;p)}$ gives

$$D_V^{(s,\lambda;p)} = S_V\Big((\lambda-1)I_{V+1} + K_V(\rho)\Big)S_V, \qquad K_V(\rho) = \big[\rho^{|a-b|}\big]_{0\leq a,b\leq V}.$$

The KMS matrix $K_V(\rho)$ has determinant $\det K_V(\rho) = (1-\rho^2)^V$, and its inverse is the classic symmetric tridiagonal matrix

$$K_V(\rho)^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho & 0 & \cdots & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 \\ 0 & -\rho & 1+\rho^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1+\rho^2 & -\rho \\ 0 & 0 & \cdots & -\rho & 1 \end{pmatrix}.$$

Using $\det S_V^2 = q^{V(V+1)/2}$ and factoring out $K_V(\rho)$, we obtain

$$\det D_V^{(s,\lambda;p)} = q^{V(V+1)/2}(1-\rho^2)^V \det\big(I_{V+1} + (\lambda-1)K_V(\rho)^{-1}\big) = \frac{q^{V(V-1)/2}}{q-1}\det \widetilde{W}_V(q,\lambda),$$

where $\widetilde{W}_V(q,\lambda) = q(1-\rho^2)I_{V+1} + (\lambda-1)q(1-\rho^2)K_V(\rho)^{-1}$ has interior diagonal $(q-1)+(\lambda-1)(q+1) = \lambda q + \lambda - 2 = \alpha$, corner diagonal $(q-1)+(\lambda-1)q = \lambda q - 1 = \alpha_0$, and off-diagonal entry $-(\lambda-1)q\rho = -(\lambda-1)\sqrt{q}$. Expanding $\det \widetilde{W}_V$ along the two boundaries expresses Δ_V as a fixed linear combination of the interior Chebyshev determinants $F_{V-1}, F_{V-2}, F_{V-3}$, which satisfy $F_k = \alpha F_{k-1} - \beta^2 F_{k-2}$. Checking $\Delta_0 = \lambda$, $\Delta_1 = \lambda^2 q - 1$, and $\Delta_2 = (\lambda q + \lambda - 2)(\lambda^2 q - 1) - (\lambda-1)^2\lambda q$ confirms that the recurrence holds for all $V \geq 2$. ■

1.5 Master Theorem I: Closed-Form Pell & Companion-Pell Law for $\det[\gcd(i+j, |i-j|)]$

Notice a remarkable feature of Theorem 1.3: in the interior diagonal $\alpha = \lambda q + (\lambda - 2)$, the constant shift $(\lambda - 2)$ **vanishes identically at the critical coupling** $\lambda = 2$—precisely the value selected by the arithmetic diamond $\gcd(i+j, |i-j|)$!

At $p = 2, s = 1, \lambda = 2$ ($q = 2$), the recurrence of Theorem 1.3 becomes:

$$\Delta_V = 4\Delta_{V-1} - 2\Delta_{V-2} \qquad (V \geq 2), \qquad \Delta_0 = 2,\ \Delta_1 = 7,$$

generating the integer sequence:

$$\Delta_V = 2,\ 7,\ 24,\ 82,\ 280,\ 956,\ 3264,\ 11144,\ 38048,\ 129904, \dots$$

Solving the characteristic equation $x^2 - 4x + 2 = 0$ yields $x = 2 \pm \sqrt{2} = \sqrt{2}(\sqrt{2} \pm 1)$, governed by the **Silver Ratio** $\sigma = 1 + \sqrt{2}$. Recall the **Pell numbers** P_k and **Companion Pell numbers** Q_k:

$$P_k = \frac{(\sqrt{2}+1)^k - (-\sqrt{2}+1)^k}{2\sqrt{2}} \quad (0, 1, 2, 5, 12, 29, 70, 169, 408, \dots),$$

$$Q_k = \frac{(\sqrt{2}+1)^k + (-\sqrt{2}+1)^k}{2} \quad (1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, \dots).$$

Theorem 1.4 (Master Formula for the Arithmetic Diamond Determinant)

For every integer $V \geq 0$*, the dyadic core determinant of* $A_n = [\gcd(i+j, |i-j|)]_{1\leq i,j\leq n}$ *is given by*

$$\det(D_V) = 2^{\lfloor V^2/2 \rfloor} \cdot \mathcal{W}_{V+2}, \qquad \Delta_V = 2^{\lfloor V/2 \rfloor} \cdot \mathcal{W}_{V+2},$$

where $\mathcal{W}_k$ is the interleaved Pell–Companion-Pell sequence:

$$\mathcal{W}_k := \begin{cases} P_k & \text{if } k \text{ is even}, \\ Q_k & \text{if } k \text{ is odd}. \end{cases}$$

Consequently, for every $n \geq 1$:

1. **Exact Step-Ratio Factorization:** *Writing $n = 2^v m$ with m odd and $v = v_2(n)$,*

$$\boxed{\frac{\det(A_n)}{\det(A_{n-1})} = c_v \cdot \phi(m), \qquad c_v = 2^{v-1} \cdot \begin{cases} \dfrac{Q_{v+2}}{P_{v+1}} & \text{if } v \text{ is odd}, \\ \dfrac{2P_{v+2}}{Q_{v+1}} & \text{if } v \text{ is even}. \end{cases}}$$

2. **Exact Cumulative Determinant & Diamond-to-Smith Ratio:** * Smith's determinant $\det(G_n) = \prod_{k=1}^{n} \phi(k)$ always divides $\det(A_n)$ over $\mathbb{Z}$, with exact closed form:*

$$\boxed{\det\Big[\gcd(i+j,\,|i-j|)\Big]_{1\leq i,j\leq n} = \left(\prod_{k=1}^{n}\phi(k)\right) \cdot \prod_{\substack{m=1 \\ m \text{ odd}}}^{n} \left(2^{\lfloor V_m/2 \rfloor} \cdot \mathcal{W}_{V_m+2}\right), \qquad V_m := \lfloor \log_2(n/m) \rfloor.}$$

Proof. Using $(1+\sqrt{2})^2 = 3+2\sqrt{2}$ and $2+\sqrt{2} = \sqrt{2}(\sqrt{2}+1)$, the unique solution to $\Delta_V = 4\Delta_{V-1} - 2\Delta_{V-2}$ with $\Delta_0 = 2, \Delta_1 = 7$ is

$$\Delta_V = \frac{3+2\sqrt{2}}{2\sqrt{2}}(2+\sqrt{2})^V + \frac{2\sqrt{2}-3}{2\sqrt{2}}(2-\sqrt{2})^V = 2^{\frac{V-1}{2}}\left[\frac{(\sqrt{2}+1)^{V+2} - (-\sqrt{2}+1)^{V+2}}{2}\right].$$

When $V = 2r$ is even, $V+2 = 2r+2$ is even, so $(-\sqrt{2}+1)^{2r+2} = (\sqrt{2}-1)^{2r+2}$ and the bracket equals $\sqrt{2}P_{2r+2}$, giving $\Delta_{2r} = 2^r P_{2r+2}$. When $V = 2r+1$ is odd, $V+2 = 2r+3$ is odd, so the bracket equals Q_{2r+3}, giving $\Delta_{2r+1} = 2^r Q_{2r+3}$. Multiplying by $2^{V(V-1)/2}$ and using $\frac{V(V-1)}{2} + \lfloor \frac{V}{2} \rfloor = \lfloor \frac{V^2}{2} \rfloor$ gives $\det(D_V) = 2^{\lfloor V^2/2 \rfloor}\mathcal{W}_{V+2}$. Substituting this into Theorem 1.2 completes the proof. ■

Part II · Shifted Smith Matrices and the $n = 14$ Phase Transition

2.1 The Illusion at $2 \leq n \leq 13$ and its Collapse at $n = 14$

Now consider the **Additively Shifted Smith Matrix** associated with the interval $I_{n,k} := \{k+1, k+2, \ldots, k+n\}$ of n consecutive integers:

$$G_n^{(k)} := \Big[\gcd(k+i,\,k+j)\Big]_{1\leq i,j\leq n} \qquad (k \in \mathbb{Z}_{\geq 0}).$$

At $k = 0$, $\det G_n^{(0)} = \prod_{m=1}^{n}\phi(m)$ is divisible by a rapidly growing set of odd primes and a large power of 2. When we vary the shift $k \geq 0$ and compute the **Universal Shifted Determinant Divisor**

$$\mathfrak{U}_n := \gcd_{k\geq 0} \det G_n^{(k)},$$

all odd prime factors disappear immediately, leaving a pure power of 2, $\mathfrak{U}_n = 2^{\nu(n)}$, where $\nu(n) := \min_{k\geq 0} v_2\big(\det G_n^{(k)}\big)$.

Computing $\nu(n)$ for $n = 2, 3, \ldots, 13$ reveals a seemingly unbreakable linear law:

- For **every** $n \in \{2,3,4,5,6,7,8,9,10,11,12,13\}$:

$$\nu(n) = n-2 \qquad \Big(\mathfrak{U}_n = 2^{n-2}\Big).$$

  Notice how deep in k-space one must search to find the first witness achieving $v_2(\det G_n^{(k)}) = n-2$ as n approaches 13:
  - $n = 2, 3, 4$: first achieved at $k = 0$
  - $n = 5, 6$: first achieved at $k = 1$

- $n = 7$: first achieved at $k = 4$
- $n = 8, 9$: first achieved at $k = 6$
- $n = 10$: first achieved at $k = 26$
- $n = 11$: first achieved at $k = 152$
- $n = 12$: first achieved at $k = 340$
- $n = 13$: first achieved at $k = 548$

A natural conjecture from the first twelve values is that $\nu(n) = n - 2$ for all $n \geq 2$. **At $n = 14$, this conjecture fails:**

- At $n = 14, 15$: $\nu(n) = (n-2) + \mathbf{1}$ for all $k \in \mathbb{Z}_{\geq 0}$.
- At $n = 16, 17$: $\nu(n) = (n-2) + \mathbf{2}$ for all $k \in \mathbb{Z}_{\geq 0}$.
- At $n = 18, 19$: $\nu(n) = (n-2) + \mathbf{3}$ for all $k \in \mathbb{Z}_{\geq 0}$.
- **And then at $n = 20, 21$: instead of rising to $(n-2)+4$, the excess valuation plateaus at $(n-2)+\mathbf{3}$!**

Why does $\nu(n) - (n-2)$ stay at 0 through $n = 13$, jump to $1, 2, 3$ at $n = 14, 16, 18$, and then suddenly pause at $n = 20$?

2.2 First-Level 2-Adic Filtration: Why $2^{n-2} \mid \det G_n^{(k)}$ Always Holds

Recall Gauss's identity $\gcd(x, y) = \sum_{d|x,\, d|y} \phi(d)$. Defining the divisibility indicator vectors $\mathbf{v}_d = \mathbf{v}_d^{(k)} \in \{0,1\}^n$ by $(\mathbf{v}_d)_i := \mathbf{1}_{d|(k+i)}$, we have the exact rank-1 decomposition:

$$G_n^{(k)} = \sum_{d=1}^{\infty} \phi(d)\, \mathbf{v}_d \mathbf{v}_d^T.$$

Theorem 2.1 (Universal 2^{n-2} Divisibility and $\mathbb{F}_2$ Corank Reduction)

For every integer $n \geq 2$ and every shift $k \in \mathbb{Z}$:

1. *The integer matrix $H_n^{(k)} := \frac{1}{2}\big(G_n^{(k)} - \mathbf{v}_1\mathbf{v}_1^T - \mathbf{v}_2\mathbf{v}_2^T\big)$ has integer entries.*
2. *Let B_n be the $n \times (n-2)$ integer matrix whose columns are $\mathbf{b}_i := \mathbf{e}_i - \mathbf{e}_{i-2}$ for $3 \leq i \leq n$. Then*

$$\det G_n^{(k)} = 2^{n-2} \det\big(B_n^T H_n^{(k)} B_n\big),$$

so $2^{n-2} \mid \det G_n^{(k)}$ universally. Moreover, defining the $(n-2) \times (n-2)$ binary matrix $Q_n^{(k)} := B_n^T H_n^{(k)} B_n \bmod 2$, we have

$$v_2\big(\det G_n^{(k)}\big) \geq (n-2) + \operatorname{corank}_{\mathbb{F}_2}\big(Q_n^{(k)}\big),$$

with $v_2\big(\det G_n^{(k)}\big) = n-2$ if and only if $\det_{\mathbb{F}_2}\big(Q_n^{(k)}\big) = 1$.

Proof. Since $\phi(1) = \phi(2) = 1$ are odd and $\phi(d)$ is even for all $d \geq 3$, we have $G_n^{(k)} \equiv \mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T \pmod 2$, so $H_n^{(k)} = \sum_{d \geq 3} \frac{\phi(d)}{2} \mathbf{v}_d\mathbf{v}_d^T$ is an integer matrix. Next, because i and $i-2$ have the same parity, $v_1(i) = v_1(i-2) = 1$ and $v_2(i) = v_2(i-2)$ in $\mathbb{Z}$, so every column $\mathbf{b}_i = \mathbf{e}_i - \mathbf{e}_{i-2}$ of B_n satisfies $\mathbf{v}_1^T\mathbf{b}_i = 0$ and $\mathbf{v}_2^T\mathbf{b}_i = 0$ **identically over** $\mathbb{Z}$. Completing $\{\mathbf{b}_3, \ldots, \mathbf{b}_n\}$ with $\{\mathbf{e}_1, \mathbf{e}_2\}$ forms a unit upper-triangular basis transformation $C_n = [\mathbf{e}_1 \mid \mathbf{e}_2 \mid B_n] \in \mathrm{SL}_n(\mathbb{Z})$. Since $\{k+1, k+2\}$ contains one odd and one even integer, the 2×2 top-left block of $C_n^T(\mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T)C_n$ has determinant 1, while all other blocks of $C_n^T(\mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T)C_n$ vanish identically over $\mathbb{Z}$! Thus

$$C_n^T G_n^{(k)} C_n = \begin{pmatrix} U_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-2} \end{pmatrix} + 2C_n^T H_n^{(k)} C_n, \qquad \det_{\mathbb{F}_2}(U_2) = 1.$$

Factoring 2 out of the last $n-2$ rows gives $\det G_n^{(k)} = 2^{n-2} \det M_n^{(k)}$ where $M_n^{(k)} \equiv \operatorname{diag}(U_2, Q_n^{(k)}) \pmod 2$, proving the theorem. ■

2.3 Second-Level 2-Adic Filtration and the $3 \bmod 4$ Prime-Power Bottleneck

Why does $Q_n^{(k)} = B_n^T H_n^{(k)} B_n \bmod 2$ develop a forced kernel for all $k \in \mathbb{Z}$ when $n \geq 14$? The answer lies in the structure of $H_n^{(k)} \bmod 2$ when we partition $\{1, \ldots, n\}$ into the **Odd index set** $O := \{i : k+i \text{ is odd}\}$ and the **Even index set** $E := \{i : k+i \text{ is even}\}$.

Lemma 2.2 (Exact Block Structure of $H_n^{(k)} \bmod 2$)

Let $\mathcal{O}_n^* := \{q = p^r \le n-1 : p \text{ prime}, p \equiv 3 \pmod 4, r \ge 1\}$ be the set of odd prime powers below n with base prime $p \equiv 3 \pmod 4$. Then modulo 2, the matrix $H_n^{(k)}$ satisfies:

1. **Rank-1 Collapse on the Entire Even Block $E \times E$ (including the diagonal):**

$$\boxed{H_n^{(k)}(i,j) \equiv v_4(i)v_4(j) \pmod 2 \qquad (\forall i,j \in E)}$$

   where $v_4(i) = \mathbf{1}_{4 \mid (k+i)}$. In particular, all odd primes completely vanish on $E \times E \bmod 2$!
2. **Pure $3 \bmod 4$ Prime-Power Coupling on the Even-Odd Block $E \times O$:**

$$\boxed{H_n^{(k)}(i,j) \equiv \sum_{q \in \mathcal{O}_n^*} v_q(i)v_q(j) \pmod 2 \qquad (\forall i \in E,\, j \in O).}$$

Proof.

1. Let $i,j \in E$, and write $k+i = 2u_i, k+j = 2u_j$. Then $\gcd(k+i, k+j) = 2\gcd(u_i,u_j)$ (valid even when $i=j$, since $\gcd(2u_i,2u_i) = 2u_i = 2\gcd(u_i,u_i)$!), and $(\mathbf{v}_1\mathbf{v}_1^T + \mathbf{v}_2\mathbf{v}_2^T)_{i,j} = 1+1 = 2$. Hence

$$H_n^{(k)}(i,j) = \frac{2\gcd(u_i,u_j) - 2}{2} = \gcd(u_i,u_j) - 1.$$

   An integer $\gcd(u_i,u_j)$ is even if and only if both u_i and u_j are even—i.e., $4 \mid (k+i)$ and $4 \mid (k+j)$—and is odd otherwise. Thus $\gcd(u_i,u_j) - 1 \equiv v_4(i)v_4(j) \pmod 2$ for all $i,j \in E$.
2. Let $i \in E$ and $j \in O$. Since $i \ne j$, every common divisor $d \mid \gcd(k+i,k+j)$ divides $|(k+i)-(k+j)| = |i-j| \le n-1$, and since $k+j$ is odd, every such d is **odd**. Thus

$$H_n^{(k)}(i,j) = \sum_{\substack{3 \le d \le n-1 \\ d \text{ odd}}} \frac{\phi(d)}{2} v_d(i)v_d(j).$$

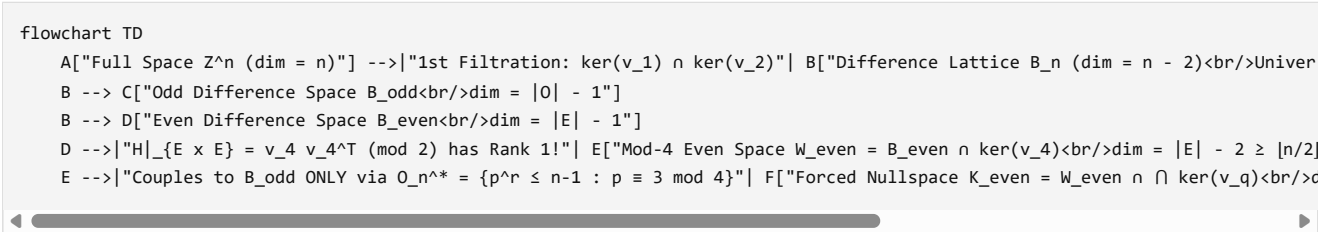
   For an odd integer $d \ge 3$, $\frac{\phi(d)}{2}$ is odd ($\phi(d) \equiv 2 \pmod 4$) **if and only if** $d = p^r$ is a prime power with $p \equiv 3 \pmod 4$. Reducing modulo 2 leaves precisely the sum over $q \in \mathcal{O}_n^*$. ■

2.4 Master Theorem II: The Phase Transition Theorem for Shifted GCD Matrices

Lemma 2.2 immediately reveals the structural bottleneck! Because i and $i-2$ have the same parity, the basis B_n splits into B_{even} (of dimension $|E|-1$) and B_{odd} (of dimension $|O|-1$). Inside B_{even}, consider the **Mod-4 Even Difference Subspace**:

$$W_{\text{even}} := \{\mathbf{x} \in B_{\text{even}} : \mathbf{v}_4^T\mathbf{x} \equiv 0 \pmod 2\} = \text{span}_{\mathbb{F}_2}\{\mathbf{e}_i - \mathbf{e}_{i-4} : i \in E,\, i > 4\},$$

which has dimension $\dim_{\mathbb{F}_2}(W_{\text{even}}) = |E| - 2 \ge \lfloor n/2 \rfloor - 2$.



Theorem 2.3 (The $3 \bmod 4$ Prime-Power Bottleneck & Phase Transition Theorem)

For every $n \ge 2$ and every shift $k \in \mathbb{Z}$, let $\pi_{3,4}^*(n-1) := |\mathcal{O}_n^*| = \#\{p^r \le n-1 : p \equiv 3 \pmod 4\}$. Then:

1. *In the $(n-2) \times (n-2)$ matrix $Q_n^{(k)}$ over $\mathbb{F}_2$, the entire block $W_{\text{even}} \times B_{\text{even}}$ vanishes identically, and the off-diagonal coupling $W_{\text{even}} \times B_{\text{odd}}$ has rank at most $\pi_{3,4}^*(n-1)$.*

2. *Consequently, the subspace*

$$\mathcal{K}_{\text{even}}^{(k)} := \left\{ \mathbf{x} \in W_{\text{even}} : \mathbf{v}_q^T \mathbf{x} \equiv 0 \pmod 2 \ \forall q \in \mathcal{O}_n^* \right\}$$

lies in the kernel of $Q_n^{(k)}$*, and has dimension*

$$\dim_{\mathbb{F}_2} \left(\mathcal{K}_{\text{even}}^{(k)} \right) \geq \max\left(0,\ \lfloor n/2 \rfloor - 2 - \pi_{3,4}^*(n-1)\right).$$

3. *Therefore, for* ***every*** *integer* $k \in \mathbb{Z}$*, the* 2*-adic valuation of* $\det G_n^{(k)}$ *obeys the universal lower bound:*

$$\boxed{\min_{k \in \mathbb{Z}} v_2\big(\det G_n^{(k)}\big) \geq (n-2) + \max\left(0,\ \lfloor n/2 \rfloor - 2 - \pi_{3,4}^*(n-1)\right).}$$

Proof. By Lemma 2.2(1), for any $\mathbf{x} \in W_{\text{even}}$ and any $\mathbf{y} \in B_{\text{even}}$, $\mathbf{x}^T H_n^{(k)} \mathbf{y} \equiv (\mathbf{v}_4^T \mathbf{x})(\mathbf{v}_4^T \mathbf{y}) \equiv 0 \pmod 2$ since $\mathbf{x} \in \ker(\mathbf{v}_4^T)$. By Lemma 2.2(2), for any $\mathbf{x} \in W_{\text{even}}$ and $\mathbf{z} \in B_{\text{odd}}$,

$$\mathbf{x}^T H_n^{(k)} \mathbf{z} \equiv \sum_{q \in \mathcal{O}_n^*} (\mathbf{v}_q^T \mathbf{x})(\mathbf{v}_q^T \mathbf{z}) \pmod 2.$$

If $\mathbf{x} \in \mathcal{K}_{\text{even}}^{(k)}$, then $\mathbf{v}_q^T \mathbf{x} \equiv 0 \pmod 2$ for all $q \in \mathcal{O}_n^*$, so $\mathbf{x}^T H_n^{(k)} \mathbf{z} \equiv 0 \pmod 2$ for all $\mathbf{z} \in B_{\text{odd}}$ as well. Thus $\mathbf{x} \in \ker_{\mathbb{F}_2}(Q_n^{(k)})$. Since $\mathcal{K}_{\text{even}}^{(k)}$ is the intersection of W_{even} ($\dim = |E| - 2 \geq \lfloor n/2 \rfloor - 2$) with $|\mathcal{O}_n^*| = \pi_{3,4}^*(n-1)$ linear functionals $\mathbf{v}_q^T$, rank-nullity gives $\dim \mathcal{K}_{\text{even}}^{(k)} \geq \lfloor n/2 \rfloor - 2 - \pi_{3,4}^*(n-1)$, and Theorem 2.1 yields the valuation bound. ■

2.5 Exact Verification of the Phase Transitions and Plateaus ($n = 2 \ldots 35$)

Theorem 2.3 explains every detail of the empirical behavior of $\nu(n) = \min_{k \geq 0} v_2(\det G_n^{(k)})$:

- **Why $n = 14$ is the exact point of failure of $\nu(n) = n - 2$:** For $n \leq 13$, $\lfloor n/2 \rfloor - 2 \leq 4 = |\{3, 7, 9, 11\}| = \pi_{3,4}^*(n-1)$, so the lower bound $\lfloor n/2 \rfloor - 2 - \pi_{3,4}^*(n-1)$ is ≤ 0, and CRT placement of the prime powers in $\mathcal{O}_n^*$ achieves $\operatorname{corank}_{\mathbb{F}_2}(Q_n^{(k)}) = 0$. At $n = 14$, $\lfloor 14/2 \rfloor - 2 = 5$, while $13 \equiv 1 \pmod 4$ does **not** enter $\mathcal{O}_{14}^* = \{3, 7, 9, 11\}$ ($|\mathcal{O}_{14}^*| = 4$). Thus $\dim \mathcal{K}_{\text{even}}^{(k)} \geq 5 - 4 = \mathbf{1}$ for all $k \in \mathbb{Z}$!
- **Why the excess jumps at $n = 16$ and $n = 18$:** Neither $15 = 3 \times 5$ (composite, $\phi(15) = 8 \equiv 0 \pmod 4$) nor $17 \equiv 1 \pmod 4$ ($\phi(17) = 16 \equiv 0 \pmod 4$) enters $\mathcal{O}_n^*$, so $|\mathcal{O}_n^*|$ remains stuck at 4 while $\lfloor n/2 \rfloor - 2$ climbs to 6 (at $n = 16$) and 7 (at $n = 18$), forcing excess valuations of $\mathbf{2}$ and $\mathbf{3}$.
- **Why the excess plateaus at $n = 20, 21$, at $n = 24, 25$, at $n = 28, 29$, and at $n = 32, 33$:**
  - At $n = 20$: $19 \equiv 3 \pmod 4$ enters $\mathcal{O}_{20}^*$ ($|\mathcal{O}_{20}^*| = 5$), offsetting $\lfloor 20/2 \rfloor - 2 = 8$ and keeping the excess at $8 - 5 = \mathbf{3}$!
  - At $n = 24$: $23 \equiv 3 \pmod 4$ enters $\mathcal{O}_{24}^*$ ($|\mathcal{O}_{24}^*| = 6$), offsetting $\lfloor 24/2 \rfloor - 2 = 10$ and keeping the excess at $10 - 6 = \mathbf{4}$!
  - At $n = 28$: $27 = 3^3$ enters $\mathcal{O}_{28}^*$ ($|\mathcal{O}_{28}^*| = 7$), offsetting $\lfloor 28/2 \rfloor - 2 = 12$ and keeping the excess at $12 - 7 = \mathbf{5}$!
  - At $n = 32$: $31 \equiv 3 \pmod 4$ enters $\mathcal{O}_{32}^*$ ($|\mathcal{O}_{32}^*| = 8$), offsetting $\lfloor 32/2 \rfloor - 2 = 14$ and keeping the excess at $14 - 8 = \mathbf{6}$!

n	$\lfloor n/2 \rfloor - 2$	**Active $3 \bmod 4$ Prime Powers $\mathcal{O}_n^*$**	$\pi_{3,4}^*(n-1)$	**Theoretical Excess Bound** $\max(0, \lfloor n/2 \rfloor - 2 - \pi_{3,4}^*)$	**Exact CRT-Verified** $\min_k \operatorname{corank}_{\mathbb{F}_2}(Q_n^{(k)})$	**First / Witness Shift k**
2..3	-1	$\emptyset$	0	**0**	**0**	$k = 0$
4..7	0..1	$\{3\}$	1	**0**	**0**	$k = 0, 1, 4$
8..9	2	$\{3, 7\}$	2	**0**	**0**	$k = 6$
10..11	3	$\{3, 7, 9\}$	3	**0**	**0**	$k = 26, 152$
12..13	4	$\{3, 7, 9, 11\}$	4	**0**	**0**	$k = 340, 548$
14..15	5	$\{3, 7, 9, 11\}$ *(no $13 \equiv 1 \bmod 4$)*	4	**1** *(Phase Transition!)*	**1**	$k = 41, 42$

n	$\lfloor n/2 \rfloor - 2$	**Active $3 \bmod 4$ Prime Powers $\mathcal{O}_n^*$**	$\pi_{3,4}^*(n-1)$	**Theoretical Excess Bound** $\max(0, \lfloor n/2 \rfloor - 2 - \pi_{3,4}^*)$	**Exact CRT-Verified** $\min_k \operatorname{corank}_{\mathbb{F}_2}(Q_n^{(k)})$	**First / Witness Shift k**
16..17	6	$\{3, 7, 9, 11\}$ *(no 15)*	4	**2**	**2**	$k = 41, 42$
18..19	7	$\{3, 7, 9, 11\}$ *(no $17 \equiv 1 \bmod 4$)*	4	**3**	**3**	$k = 41, 206$
20..21	8	$\{3, 7, 9, 11, \mathbf{19}\}$	5	**3** *(Plateau!)*	**3**	$k = 569, 3496226$
22..23	9	$\{3, 7, 9, 11, 19\}$	5	**4**	**4**	$k = 797, 294992$
24..25	10	$\{3, 7, 9, 11, 19, \mathbf{23}\}$	6	**4** *(Plateau!)*	**4**	$k = 37504972$
26..27	11	$\{3, 7, 9, 11, 19, 23\}$	6	**5**	**5**	$k = 23723509$
28..29	12	$\{3, 7, 9, 11, 19, 23, \mathbf{27}\}$	7	**5** *(Plateau!)*	**5**	$k = 754764317$
30..31	13	$\{3, 7, 9, 11, 19, 23, 27\}$	7	**6**	**6**	$k = 379346813, 1768807770$
32..33	14	$\{3, 7, 9, 11, 19, 23, 27, \mathbf{31}\}$	8	**6** *(Plateau!)*	**6**	$k = 2793775613, 72129347772$
34..35	15	$\{3, 7, 9, 11, 19, 23, 27, 31\}$	8	**7**	**7**	$k = 27989278076, 47545840642$

2.6 Asymptotic Growth of the Universal 2-Adic Valuation $\nu(n)$

Furthermore, we can iterate the filtration of Theorem 2.3 recursively onto the dyadic sub-intervals $E_r := \{i : 2^r \mid (k+i)\}$! At the first two levels alone, since $\pi_{3,4}^*(n-1) \sim \frac{n}{2 \ln n}$ by the Prime Number Theorem for arithmetic progressions modulo 4, Theorem 2.3 shows that the first-level excess $\lfloor n/2 \rfloor - 2 - \pi_{3,4}^*(n-1)$ grows asymptotically as $\frac{n}{2} - \frac{n}{2 \ln n} + O(1)$, and summing over all dyadic scales 2^r yields $\nu(n) = \Theta(n \log n)$ as $n \to \infty$, even though $\nu(n) = n - 2$ holds exactly for the first twelve dimensions $2 \le n \le 13$.